Fractional Schrödinger equation

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Some properties of the fractional Schrödinger equation are studied. We prove the Hermiticity of the fractional Hamilton operator and establish the parity conservation law for fractional quantum mechanics. As physical applications of the fractional Schrödinger equation we find the energy spectra of a hydrogenlike atom (fractional ''Bohr atom'') and of a fractional oscillator in the semiclassical approximation. An equation for the fractional probability current density is developed and discussed. We also discuss the relationships between the fractional and standard Schrödinger equations.

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I. INTRODUCTION

The Feynman path integral approach to quantum mechanics [1,2] is in fact integration over Brownian-like quantum mechanical paths. Brownian motion is a special case of a Lévy α-stable random process. In the mid-1930s Lévy and Khintchine posed the question: When does the sum of N independent identically distributed quantities $X=X_1+X_2\cdots+X_N$ have the same probability distribution $p_N(X)$ (up to a scale factor) as the individual steps $p_i(X_i)$, $i=1,...,N$? The traditional answer is that each $p_i(X_i)$ should be a Gaussian, because of the central limit theorem. In other words, a sum of N Gaussians is again a Gaussian. Lévy and Khintchine proved that there exists the possibility of generalizing the central limit theorem [3,4]. They discovered a class of non-Gaussian Lévy α-stable (stable under summation) probability distributions. Each α-stable probability distribution has a stability index α often called the Lévy index, $0<\alpha\leqslant 2$. When $\alpha=2$ the Lévy α-stable distribution is transformed to the well-known Gaussian probability distribution or, in other words, the Lévy motion is transformed to Brownian motion [5,6].

The possibility of developing the path integral over the paths of the Lévy motion was discussed by Kac [7], who pointed out that the Lévy path integral generates the functional measure in the space of left (or right) continued functions having only discontinuities of the first kind.

It was shown by Montroll [8] that for a free quantum mechanical particle the chain condition for the kernel can be solved exactly and leads in general to a kernel that is the quantum analog of the classical Lévy transition probability.[1] A recent review of Montroll's paper can be found in [9], where the fractional differential equation for a free kernel was derived.

In Refs. [10,11] it was shown that the path integral over Lévy-like quantum mechanical paths allows one to develop a generalization of quantum mechanics; namely, if the path integral over Brownian trajectories leads to the well known Schrödinger equation, then the path integral over Lévy trajectories leads to the fractional Schrödinger equation. The fractional Schrödinger equation includes a space derivative of order α instead of the second order ($\alpha=2$) space derivative in the standard Schrödinger equation. Thus, the fractional Schrödinger equation is a fractional differential equation in accordance with modern terminology (see, for example, [12–17]). This is the main point of the term ''fractional Schrödinger equation'' or of the more general term fractional quantum mechanics (FQM) [11]. As mentioned above, at $\alpha=2$ the Lévy motion becomes Brownian motion. Thus, FQM includes the standard QM as a particular Gaussian case at $\alpha=2$. The quantum mechanical path integral over the Lévy paths at $\alpha=2$ becomes the well known Feynman path integral [1,2].

The non-Gaussian path integral over the Lévy paths developed in [10,11] captures the important physical property of self-similarity of the quantum mechanical path. Self-similarity means that the whole object looks like any of its parts, and we can partition the quantum mechanical path into smaller and smaller parts and each part will have the same statistical structure. Self-similarity constrains the possible choice of the kinematic (momentum-dependent) part of the Hamilton function; namely, the term $D_\alpha|\mathbf{p}|^\alpha$ [here $\mathbf{p}$ is the particle momentum; see Eq. (2)] is in general a possible choice for the kinematic part of the Hamilton function. In the special Gaussian case when $\alpha=2$ we get the quadratic kinematic term $\mathbf{p}^2/2m$ where m is the particle mass [11].

Another type of non-Gaussian path integral is the integral over paths of a truncated Lévy distribution. For the definition of the truncated Lévy distribution see, for example, [18]. The truncated Lévy distribution is widely applied to model option pricing. The path integral over truncated Lévy paths has been developed and applied to model financial dynamics for assets with non-Gaussian price fluctuations [19] (see also [20], Chap. 20). However, the procedure of truncation destroys the α-stable property and self-similarity of the original Lévy distribution. As a result the path integral developed in [19] does not capture the important physical property of self-similarity. In addition, the ''Hamiltonian function'' resulting from the truncation procedure [see Eq. (3) of Ref. [19]] leads to a nonlocal Schrödinger-like equation [see Eq. (17) of Ref. [19]], while the standard Schrödinger equation [1,21] and the developed fractional Schrödinger equation are local quantum mechanical equations.

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[1]Thanks to M. F. Shlesinger, who pointed out Montroll's paper.

The paper is organized as follows. In Sec. II the quantum mechanical path integral over the Lévy paths is discussed and the three-dimensional (3D) fractional Schrödinger equation is derived in term of the Riesz fractional derivative. In Sec. III we prove the Hermiticity of the fractional Hamilton operator in Hilbert space with a scalar product defined in the same way as for conventional quantum mechanics. The parity conservation law for fractional quantum mechanics is established.

The time-independent fractional Schrödinger equation is introduced and its properties are studied in Sec. IV. As physical applications of the time-independent fractional Schrödinger equation we have found (i) the energy spectrum and equation for the orbit radius of a hydrogenlike atom, the fractional ''Bohr atom,'' and (ii) the energy spectrum of a 1D fractional oscillator in the semiclassical approximation. In Sec. V an equation for the fractional probability current density is derived and discussed. In the Conclusions we discuss the relationship between the fractional and the well-known Schrödinger equation.

II. PATH INTEGRAL

A. Path integral over the Lévy paths

If a particle at an initial time t_a starts from the point $\mathbf{r}_a$ and goes to a final point $\mathbf{r}_b$ at time t_b, we will say simply that the particle goes from a to b, and its path $\mathbf{r}(t)$ has the property that $\mathbf{r}(t_a)=\mathbf{r}_a$ and $\mathbf{r}(t_b)=\mathbf{r}_b$. In quantum mechanics, then, we have a quantum mechanical kernel, often called a kernel, to get from the point a to the point b. This will be the sum over all of the trajectories that go between these end points and of a contribution from each. If we have the quantum particle moving in the potential $V(\mathbf{r})$ then the fractional quantum mechanical kernel $K_L(\mathbf{r}_b t_b|\mathbf{r}_a t_a)$ may be written as [11]

$$K_L(\mathbf{r}_b t_b|\mathbf{r}_a t_a)=\int_{\mathbf{r}(t_a)=\mathbf{r}_a}^{\mathbf{r}(t_b)=\mathbf{r}_b} D\mathbf{r}(\tau)\int D\mathbf{p}(\tau) \times\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau[\mathbf{p}(\tau)\dot{\mathbf{r}}(\tau) - H_\alpha(\mathbf{p}(\tau),\mathbf{r}(\tau))]\right\}, \tag{1}$$

where $\hbar$ is Planck's constant, $\dot{\mathbf{r}}$ denotes the time derivative, $H_\alpha(\mathbf{p}(\tau),\mathbf{r}(\tau))$ is the fractional Hamiltonian given by

$$H_\alpha(\mathbf{p},\mathbf{r})=D_\alpha|\mathbf{p}|^\alpha+V(\mathbf{r}), \quad 1<\alpha\leqslant 2, \tag{2}$$

with the replacement $\mathbf{p}\to\mathbf{p}(\tau)$, $\mathbf{r}\to\mathbf{r}(\tau)$, and $\{\mathbf{p}(\tau),\mathbf{r}(\tau)\}$ is the particle trajectory in phase space. The quantity D_α has the physical dimension

$$[D_\alpha]=\mathrm{erg}^{1-\alpha}\times\mathrm{cm}^\alpha\times\mathrm{sec}^{-\alpha}.$$

The phase space path integral $\int_{\mathbf{r}(t_a)=\mathbf{r}_a}^{\mathbf{r}(t_b)=\mathbf{r}_b} D\mathbf{r}(\tau)\int D\mathbf{p}(\tau)\cdots$ in Eq. (1) is defined by

$$\int_{\mathbf{r}(t_a)=\mathbf{r}_a}^{\mathbf{r}(t_b)=\mathbf{r}_b} D\mathbf{r}(\tau)\int D\mathbf{p}(\tau)\cdots = \lim_{N\to\infty}\int_{-\infty}^{\infty} d\mathbf{r}_1\cdots d\mathbf{r}_{N-1}\frac{1}{(2\pi\hbar)^{3N}}\int_{-\infty}^{\infty} d\mathbf{p}_1\cdots d\mathbf{p}_N \times\exp\left\{i\frac{\mathbf{p}_1(\mathbf{r}_1-\mathbf{r}_a)}{\hbar}-i\frac{D_\alpha\zeta|\mathbf{p}_1|^\alpha}{\hbar}\right\}\times\cdots \times\exp\left\{i\frac{\mathbf{p}_N(\mathbf{r}_b-\mathbf{r}_{N-1})}{\hbar}-i\frac{D_\alpha\zeta|\mathbf{p}_N|^\alpha}{\hbar}\right\}\cdots, \tag{3}$$

here $\zeta=(t_b-t_a)/N$.

The exponential in Eq. (1) can be written as $\exp\{(i/\hbar)S_\alpha(\mathbf{p},\mathbf{r})\}$ if we introduce the classical mechanical action for the trajectory $\{\mathbf{p}(t),\mathbf{r}(t)\}$ in phase space:

$$S_\alpha(\mathbf{p},\mathbf{r})=\int_{t_a}^{t_b} d\tau[\mathbf{p}(\tau)\dot{\mathbf{r}}(\tau)-H_\alpha(\mathbf{p}(\tau),\mathbf{r}(\tau),\tau)]. \tag{4}$$

When $\alpha=2$, $D_\alpha=1/2m$ (m is the mass of a particle) Eq. (2) is transformed into the well known Hamiltonian with kinetic energy $\mathbf{p}^2/2m$ and Eq. (1) becomes the definition of the Feynman path integral in the phase space representation; see, for example, [20].

B. Fractional Schrödinger equation

The kernel $K_L(\mathbf{r}_b t_b|\mathbf{r}_a t_a)$ which is defined by Eq. (1) describes the evolution of the quantum mechanical system

$$\psi_f(\mathbf{r}_b,t_b)=\int d\mathbf{r}_a K_L(\mathbf{r}_b t_b|\mathbf{r}_a t_a)\psi_i(\mathbf{r}_a,t_a), \tag{5}$$

where $\psi_i(\mathbf{r}_a,t_a)$ is the wave function of the initial state (at $t=t_a$) and $\psi_f(\mathbf{r}_b,t_b)$ is the wave function of the final state (at $t=t_b$).

In order to obtain the differential equation for the wave function $\psi(\mathbf{r},t)$ we apply Eq. (5) in the special case that the time differs only by an infinitesimal interval ϵ from t_a:

$$\psi(\mathbf{r},t+\epsilon)=\int d\mathbf{r}' K_L(\mathbf{r},t+\epsilon|\mathbf{r}',t)\psi(\mathbf{r}',t).$$

Using Feynman's approximation $\int_t^{t+\epsilon} d\tau\, V(\mathbf{r}(\tau),\tau) \simeq \epsilon V((\mathbf{r}+\mathbf{r}')/2,t)$ and the definition given by Eq. (1) we have

$$\psi(\mathbf{r},t+\epsilon)=\int d\mathbf{r}'\frac{1}{(2\pi\hbar)^3}\int_{-\infty}^{\infty} d\mathbf{p}\exp\left\{i\frac{\mathbf{p}(\mathbf{r}'-\mathbf{r})}{\hbar} - i\frac{D_\alpha\epsilon|\mathbf{p}|^\alpha}{\hbar}-\frac{i}{\hbar}\epsilon V\left(\frac{\mathbf{r}+\mathbf{r}'}{2},t\right)\right\}\psi(\mathbf{r}',t).$$

We may expand the left-hand and the right-hand sides in power series:

$$\psi(\mathbf{r},t)+\epsilon\frac{\partial\psi(\mathbf{r},t)}{\partial t}=\int d\mathbf{r}'\frac{1}{(2\pi\hbar)^3}\int_{-\infty}^{\infty}d\mathbf{p}\times\exp\left\{i\frac{\mathbf{p}(\mathbf{r}'-\mathbf{r})}{\hbar}\right\}\left(1-i\frac{D_\alpha\epsilon|\mathbf{p}|^\alpha}{\hbar}\right)\times\left[1-\frac{i}{\hbar}\epsilon V\left(\frac{\mathbf{r}+\mathbf{r}'}{2},t\right)\right]\psi(\mathbf{r}',t). \quad (6)$$

Then, taking into account the definitions of the Fourier transforms

$$\psi(\mathbf{r},t)=\frac{1}{(2\pi\hbar)^3}\int d\mathbf{p}\,e^{ipx/\hbar}\varphi(\mathbf{p},t),$$

$$\varphi(\mathbf{p},t)=\int d\mathbf{p}\,e^{-ipx/\hbar}\psi(\mathbf{r},t),$$

and introducing the 3D quantum Riesz fractional derivative[2] $(-\hbar^2\Delta)^{\alpha/2}$,

$$(-\hbar^2\Delta)^{\alpha/2}\psi(\mathbf{r},t)=\frac{1}{(2\pi\hbar)^3}\int d^3p\,e^{ipr/\hbar}|\mathbf{p}|^\alpha\varphi(\mathbf{p},t), \quad (7)$$

(here $\Delta=\partial^2/\partial\mathbf{r}^2$ is the Laplacian), we obtain from Eq. (6)

$$\psi(\mathbf{r},t)+\epsilon\frac{\partial\psi(\mathbf{r},t)}{\partial t}=\psi(\mathbf{r},t)-i\frac{D_\alpha\epsilon}{\hbar}(-\hbar^2\Delta)^{\alpha/2}\psi(\mathbf{r},t)-\frac{i}{\hbar}\epsilon V(\mathbf{r},t)\psi(\mathbf{r},t).$$

This will be true to order ϵ if $\psi(\mathbf{r},t)$ satisfies the differential equation

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}=D_\alpha(-\hbar^2\Delta)^{\alpha/2}\psi(\mathbf{r},t)+V(\mathbf{r},t)\psi(\mathbf{r},t). \quad (8)$$

This is the fractional Schrödinger equation. The space derivative in this equation is of fractional (noninteger) order α.

The above consideration is in fact a generalization of the well-known Feynman approach to reduce the path integral to a differential equation [1,2].

Equation (8) may be rewritten in the operator form, namely,

$$i\hbar\frac{\partial\psi}{\partial t}=H_\alpha\psi, \quad (9)$$

where H_α is the fractional Hamiltonian operator

$$H_\alpha=D_\alpha(-\hbar^2\Delta)^{\alpha/2}+V(\mathbf{r},t). \quad (10)$$

By definition (10) and introducing the momentum operator $\mathbf{p}=i\hbar\nabla$ one may obtain the fractional Hamiltonian H_α in the form given by Eq. (2).

[2]The Riesz fractional derivative was originally introduced in [22].

Since the kernel $K_L(\mathbf{r}_bt_b|\mathbf{r}_at_a)$ thought of as a function of variables $\mathbf{r}_b,t_b$ is a special wave function (namely, that for a particle which starts at $\mathbf{r}_a,t_a$), we see that K_L must also satisfy a fractional Schrödinger equation. Thus for the quantum system described by the fractional Hamiltonian Eq. (10) we have

$$i\hbar\frac{\partial}{\partial t_b}K_L(\mathbf{r}_bt_b|\mathbf{r}_at_a)=D_\alpha(-\hbar^2\Delta_b)^{\alpha/2}K_L(\mathbf{r}_bt_b|\mathbf{r}_at_a)+V(\mathbf{r}_b,t)K_L(\mathbf{r}_bt_b|\mathbf{r}_at_a), \quad (11)$$

where $t_b>t_a$, and the subscript b on Δ_b means that the fractional derivative acts on the variable $\mathbf{r}_b$.

III. QUANTUM RIESZ FRACTIONAL DERIVATIVE

A. Hermiticity of the fractional Hamilton operator

The fractional Hamiltonian H_α given by Eq. (10) is the Hermitian operator in the space with scalar product

$$(\phi,\chi)=\int_{-\infty}^{\infty}d\mathbf{r}\,\phi^*(\mathbf{r},t)\chi(\mathbf{r},t), \quad (12)$$

where the superscript * means as usual the complex conjugate.

To prove the Hermiticity of the fractional Hamilton H_α let us note that in accordance with the definition of the quantum Riesz fractional derivative given by Eq. (7) there exists the formula for integration by parts

$$(\phi,(-\hbar^2\Delta)^{\alpha/2}\chi)=((-\hbar^2\Delta)^{\alpha/2}\phi,\chi). \quad (13)$$

The average energy of a fractional quantum system with Hamiltonian H_α is

$$E_\alpha=\int_{-\infty}^{\infty}d\mathbf{r}\,\psi^*(\mathbf{r},t)H_\alpha\psi(\mathbf{r},t). \quad (14)$$

Taking into account Eq. (13) we have

$$E_\alpha=\int_{-\infty}^{\infty}d\mathbf{r}\,\psi^*(\mathbf{r},t)H_\alpha\psi(\mathbf{r},t)=\int_{-\infty}^{\infty}d\mathbf{r}(H_\alpha^\dagger\psi(\mathbf{r},t))^*\psi(\mathbf{r},t)=E_\alpha^*,$$

and as a physical consequence the energy of the system is real. Thus, the fractional Hamiltonian H_α defined by Eq. (10) is a Hermitian or self-adjoint operator in the space with the scalar product defined by Eq. (12):

$$(H_\alpha^\dagger\phi,\chi)=(\phi,H_\alpha\chi). \quad (15)$$

B. Parity conservation law for fractional quantum mechanics

It follows from the definition (7) of the quantum Riesz fractional derivative that

$$(-\hbar^2\Delta)^{\alpha/2}\exp\left\{i\frac{\mathbf{p}\cdot x}{\hbar}\right\}=|\mathbf{p}|^{\alpha}\exp\left\{i\frac{\mathbf{p}\cdot x}{\hbar}\right\}. \tag{16}$$

Thus, the function $\exp\{i\mathbf{p}\cdot\mathbf{x}/\hbar\}$ is the eigenfunction of the 3D quantum Riesz fractional operator $(-\hbar^2\Delta)^{\alpha/2}$ with eigenvalue $|\mathbf{p}|^{\alpha}$.

The operator $(-\hbar^2\Delta)^{\alpha/2}$ is a symmetrized fractional derivative, that is,

$$(-\hbar^2\Delta_{\mathbf{r}})^{\alpha/2}\cdots=(-\hbar^2\Delta_{-\mathbf{r}})^{\alpha/2}\cdots. \tag{17}$$

Because of the property (17) the fractional Hamiltonian H_{α} given by Eq. (10) remains invariant under *inversion* transformation. Inversion, or, to be precise, spatial inversion, consists in the simultaneous change of sign of all three spatial coordinates:

$$\mathbf{r}\rightarrow-\mathbf{r},\quad x\rightarrow-x,\quad y\rightarrow-y,\quad z\rightarrow-z. \tag{18}$$

Let us denote the inversion operator by $\hat{P}$. The inverse symmetry is the fact that $\hat{P}$ and the fractional Hamiltonian H_{α} commute,

$$\hat{P}H_{\alpha}=H_{\alpha}\hat{P}. \tag{19}$$

We can divide the wave functions of quantum mechanical states with a well-defined eigenvalue of the operator $\hat{P}$ into two classes: (i) functions that are not changed when acted upon by the inversion operator,

$$\hat{P}\psi_{+}(\mathbf{r})=\psi_{+}(\mathbf{r})$$

(the corresponding states are called even states); and (ii) functions that change sign under the action of the inversion operator,

$$\hat{P}\psi_{-}(\mathbf{r})=-\psi_{-}(\mathbf{r})$$

(the corresponding states are called odd states).

Equation (19) expresses the ''parity conservation law;'' if the state of a closed fractional quantum mechanical system has a given parity (i.e., if it is even or odd), then this parity is conserved.

IV. TIME-INDEPENDENT FRACTIONAL SCHRÖDINGER EQUATION

The special case when the Hamiltonian H_{α} does not depend explicitly on the time is of great importance for physical applications. It is easy to see that in this case there exists a special solution of the fractional Schrödinger equation (8) of the form

$$\psi(\mathbf{r},t)=e^{-(i/\hbar)Et}\phi(\mathbf{r}), \tag{20}$$

where $\phi(\mathbf{r})$ satisfies

$$H_{\alpha}\phi(\mathbf{r})=E\phi(\mathbf{r}), \tag{21}$$

or

$$D_{\alpha}(-\hbar^2\Delta)^{\alpha/2}\phi(\mathbf{r})+V(\mathbf{r})\phi(\mathbf{r})=E\phi(\mathbf{r}),$$

$$1<\alpha\leqslant 2. \tag{22}$$

We call Eq. (22) the time-independent (or stationary) fractional Schrödinger equation.

A. Fractional Bohr atom

The hydrogenlike potential energy $V(\mathbf{r})$ is

$$V(\mathbf{r})=-\frac{Ze^2}{|\mathbf{r}|}.$$

Then the fractional Schrödinger equation (22) has the form

$$D_{\alpha}(-\hbar^2\Delta)^{\alpha/2}\phi(\mathbf{r})-\frac{Ze^2}{|\mathbf{r}|}\phi(\mathbf{r})=E\phi(\mathbf{r}), \tag{23}$$

and can be treated as fractional eigenvalue problem.

The total energy of the considered quantum mechanical system is

$$E=E_{\mathrm{kin}}+V,$$

where E_{kin} is the kinetic energy

$$E_{\mathrm{kin}}=D_{\alpha}|\mathbf{p}|^{\alpha}, \tag{24}$$

and V is the potential energy

$$V=-\frac{Ze^2}{|\mathbf{r}|}. \tag{25}$$

It is well known that if the potential energy is a homogeneous function of the coordinates and the motion takes place in a finite region of space, there exists a simple relation between the time averaged values of the kinetic and potential energies, known as the *virial theorem* (see [23], p. 23). It follows from the virial theorem that between the average kinetic energy (24) and the average potential energy (25) there exists the relation

$$\alpha\overline{E}_{\mathrm{kin}}=-\overline{V}, \tag{26}$$

where the average value $\overline{f}$ of any function of time is defined as

$$\overline{f}=\lim_{T\rightarrow\infty}\frac{1}{T}\int_0^{\infty}dt\,f(t).$$

In order to evaluate the energy spectrum of the fractional hydrogenlike atom let us recall the *Niels Bohr postulates* [24].

(1) The electron moves in orbits restricted by the requirement that the angular momentum be an integral multiple of $\hbar$, that is, for circular orbits of radius a_n, the electron momentum is restricted by

$$pa_n=n\hbar\quad(n=1,2,3,\ldots), \tag{27}$$

and furthermore the electrons in these orbits do not radiate in spite of their acceleration. They are said to be in stationary states.

(2) Electrons can make discontinuous transitions from one allowed orbit corresponding to $n=n_2$ to another corresponding to $n=n_1$, and the change in energy will appear as radiation with frequency

$$\omega=\frac{E_{n2}-E_{n1}}{\hbar}, \quad (n_2>n_1). \tag{28}$$

An atom may absorb radiation by having its electrons make a transition to a higher energy orbit.

Using the first Bohr postulate and Eq. (26) yields

$$\alpha D_\alpha\left(\frac{n\hbar}{a_n}\right)^\alpha=\frac{Ze^2}{a_n},$$

from which follows the equation for the radius of the fractional Bohr orbits

$$a_n=a_0 n^{\alpha/(\alpha-1)}, \tag{29}$$

here a_0 is the fractional Bohr radius (the radius of the lowest, $n=1$ Bohr orbit) defined as

$$a_0=\left(\frac{\alpha D_\alpha \hbar^\alpha}{Ze^2}\right)^{1/(\alpha-1)}. \tag{30}$$

By using Eq. (26) we find for the total average energy $\overline{E}$,

$$\overline{E}=(1-\alpha)\overline{E}_{\rm kin}.$$

Thus, for the energy levels of the fractional hydrogenlike atom we have

$$E_n=-(\alpha-1)E_0 n^{-\alpha/(\alpha-1)}, \quad 1<\alpha\leqslant 2, \tag{31}$$

where E_0 is the binding energy of the electron in the lowest Bohr orbit, that is, the energy required to put it in a state with $E=0$ corresponding to $n=\infty$,

$$E_0=\left(\frac{(Ze^2)^\alpha}{\alpha^2 D_\alpha \hbar^\alpha}\right)^{1/(\alpha-1)}. \tag{32}$$

The energy $(\alpha-1)E_0$ can be considered as a generalization of the Rydberg constant of standard quantum mechanics. It is easy to see that at $\alpha=2$ the energy $(\alpha-1)E_0$ is transformed into the well-known expression for the Rydberg constant, $\mathrm{Ry}=me^4/2\hbar^2$.

The frequency of the radiation ω associated with the transition, say, for example from k to n, $k\to n$, is

$$\omega=\frac{(\alpha-1)E_0}{\hbar}\left[\frac{1}{n^{\alpha/(\alpha-1)}}-\frac{1}{k^{\alpha/(\alpha-1)}}\right] \quad (k>n). \tag{33}$$

Equations (29)–(33) give a generalization of the Bohr atom theory. In the special Gaussian case, $\alpha=2$ (standard quantum mechanics), Eqs. (29)–(33) reproduce the well-known results of the Bohr theory [24,25]. The existence of Eqs. (29)–(33) is a result of deviation of the fractal dimension $d_{\rm fractal}^{(\text{Lévy})}$ of the Lévy-like quantum mechanical path from 2, $d_{\rm fractal}^{(\text{Lévy})}=\alpha<2$.

B. Spectrum of the 1D fractional oscillator in the semiclassical approximation

The fractional oscillator introduced in [10] is the model with the fractional Hamiltonian operator $H_{\alpha,\beta}$,

$$H_{\alpha,\beta}=D_\alpha(-\hbar^2\Delta)^{\alpha/2}+q^2|\mathbf{r}|^\beta,$$

$$1<\alpha\leqslant 2, \quad 1<\beta\leqslant 2, \tag{34}$$

where $\mathbf{r}$ is the 3D vector, $\Delta=\partial^2/\partial\mathbf{r}^2$ is the Laplacian, the operator $(-\hbar^2\Delta)^{\alpha/2}$ is defined by Eq. (7) and q is a constant with physical dimension $[q]=\mathrm{erg}^{1/2}\times\mathrm{cm}^{-\beta/2}$.

The 1D fractional oscillator with the Hamilton function $H_{\alpha,\beta}=D_\alpha|p|^\alpha+q^2|x|^\beta$ poses an interesting problem for semiclassical treatment. We set the total energy equal to E, so that

$$E=D_\alpha|p|^\alpha+q^2|x|^\beta, \tag{35}$$

whence

$$|p|=\left(\frac{1}{D_\alpha}(E-q^2|x|^\beta)\right)^{1/\alpha}.$$

At the turning points $p=0$. Thus, classical motion is possible in the range $|x|\leqslant(E/q^2)^{1/\beta}$.

A routine use of the Bohr-Sommerfeld quantization rule [21] yields

$$2\pi\hbar\left(n+\frac{1}{2}\right)=\oint p\,dx=4\int_0^{x_m}p\,dx$$

$$=\frac{4}{D_\alpha^{1/\alpha}}\int_0^{x_m}(E-q^2|x|^\beta)^{1/\alpha}dx, \tag{36}$$

where the notation $\oint$ means the integral over one complete period of the classical motion, and $x_m=(E/q^2)^{1/\beta}$ is the turning point of classical motion. To evaluate the integral on the right hand side of Eq. (36) we introduce a new variable $y=x(E/q^2)^{-1/\beta}$. Then we have

$$\int_0^{x_m}(E-q^2|x|^\beta)^{1/\alpha}dx=\frac{1}{q^{2/\beta}}E^{1/\alpha+1/\beta}\int_0^1 dy(1-y^\beta)^{1/\alpha}. \tag{37}$$

The integral over dy can be expressed in terms of the B function.[3] Indeed, the substitution $z=y^\beta$ yields

[3]The $B(a,b)$ function has the familiar integral representation [26] $B(a,b)=\int_0^1 du\,u^{a-1}(1-u)^{b-1}$.

$$\int_0^1 dy(1-y^\beta)^{1/\alpha}=\frac{1}{\beta}\int_0^1 dz\, z^{1/\beta-1}(1-z)^{1/\alpha}$$

$$=\frac{1}{\beta}B\left(\frac{1}{\beta},\frac{1}{\alpha}+1\right). \tag{38}$$

With the help of Eqs. (37) and (38) we rewrite Eq. (36) as

$$2\pi\hbar\left(n+\frac{1}{2}\right)=\frac{4}{D_\alpha^{1/\alpha}q^{2/\beta}}E^{1/\alpha+1/\beta}\frac{1}{\beta}B\left(\frac{1}{\beta},\frac{1}{\alpha}+1\right).$$

The above equation gives the values of the energies of stationary states for a 1D fractional oscillator,

$$E_n=\left(\frac{\pi\hbar\beta D_\alpha^{1/\alpha}q^{2/\beta}}{2B(1/\beta,1/\alpha+1)}\right)^{\alpha\beta/(\alpha+\beta)}\left(n+\frac{1}{2}\right)^{\alpha\beta/(\alpha+\beta)}. \tag{39}$$

This equation generalizes the well-known energy spectrum of the standard quantum mechanical oscillator (see, for example, [21]) and is transformed to it in the special case $\alpha=2$, $\beta=2$.

It follows from Eq. (39) that at

$$\frac{\alpha\beta}{\alpha+\beta}=1 \tag{40}$$

the energy spectrum becomes equidistant. When $1<\alpha\leqslant 2$ and $1<\beta\leqslant 2$ the condition given by Eq. (40) occurs for $\alpha=2$ and $\beta=2$ only. This means that only the standard quantum mechanical oscillator has an equidistant energy spectrum.

V. CURRENT DENSITY

By multiplying Eq. (8) from the left by $\psi^*(\mathbf{r},t)$ and the conjugate complex of Eq. (8) by $\psi(\mathbf{r},t)$ and subtracting the two resulting equations, we finally obtain

$$\frac{\partial}{\partial t}\int d^3r[\psi^*(\mathbf{r},t)\psi(\mathbf{r},t)]$$

$$=\frac{D_\alpha}{i\hbar}\int d^3r[\psi^*(\mathbf{r},t)(-\hbar^2\Delta)^{\alpha/2}\psi(\mathbf{r},t)$$

$$-\psi(\mathbf{r},t)(-\hbar^2\Delta)^{\alpha/2}\psi^*(\mathbf{r},t)]. \tag{41}$$

From this integral relationship we are led to the following well-known differential equation:

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t}+\operatorname{div}\mathbf{j}(\mathbf{r},t)=0, \tag{42}$$

where

$$\rho(\mathbf{r},t)=\psi^*(\mathbf{r},t)\psi(\mathbf{r},t) \tag{43}$$

is the probability density and the vector $\mathbf{j}(\mathbf{r},t)$ can be called the fractional probability current density vector:

$$\mathbf{j}(\mathbf{r},t)=\frac{D_\alpha\hbar}{i}[\psi^*(\mathbf{r},t)(-\hbar^2\Delta)^{\alpha/2-1}\nabla\psi(\mathbf{r},t)$$

$$-\psi(\mathbf{r},t)(-\hbar^2\Delta)^{\alpha/2-1}\nabla\psi^*(\mathbf{r},t)], \tag{44}$$

where we use the notation

$$\nabla=\frac{\partial}{\partial\mathbf{r}}.$$

Introducing the momentum operator $\hat{\mathbf{p}}=(\hbar/i)\nabla$ we can write the vector $\mathbf{j}$ in the form

$$\mathbf{j}=D_\alpha[\psi(\hat{\mathbf{p}}^2)^{\alpha/2-1}\hat{\mathbf{p}}\psi^*+\psi^*(\hat{\mathbf{p}}^{*2})^{\alpha/2-1}\hat{\mathbf{p}}^*\psi],$$

$$1<\alpha\leqslant 2. \tag{45}$$

When $\alpha=2$, $D_\alpha=1/2m$ Eqs. (44) and (45) become the well-known equations of standard quantum mechanics (see, for example, [21]). Thus we conclude that Eqs. (44) and (45) are the fractional generalization of the well-known equations for the probability current density vector of standard quantum mechanics.

To this end, we express Eq. (45) in terms of the velocity operator, which is defined as usual:

$$\hat{\mathbf{v}}=\frac{d}{dt}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the operator of the coordinates. Using the general quantum mechanical rule for differentiation of the operator,

$$\frac{d}{dt}\hat{\mathbf{r}}=\frac{i}{\hbar}[H_\alpha,\mathbf{r}],$$

we have

$$\hat{\mathbf{v}}=\frac{i}{\hbar}(H_\alpha\mathbf{r}-\mathbf{r}H_\alpha).$$

Further, with the help of the equation

$$f(\hat{\mathbf{p}})\mathbf{r}-\mathbf{r}f(\hat{\mathbf{p}})=-i\hbar\frac{\partial f}{\partial\mathbf{p}};$$

which holds for any function $f(\hat{\mathbf{p}})$ of the momentum operator, and taking into account Eq. (2) for the Hamiltonian we find the equation for the velocity operator

$$\hat{\mathbf{v}}=\alpha D_\alpha|\hat{\mathbf{p}}^2|^{\alpha/2-1}\hat{\mathbf{p}}, \tag{46}$$

here $\hat{\mathbf{p}}$ is the momentum operator. By comparing Eqs. (45) and (46) we finally conclude that

$$\mathbf{j}=\frac{1}{\alpha}(\psi\hat{\mathbf{v}}\psi^*+\psi^*\hat{\mathbf{v}}\psi),\quad 1<\alpha\leqslant 2. \tag{47}$$

To get the probability current density equal to 1 (the current when one particle passes through unit area per unit time) the wave function of the free particle has to be normalized as follows:

$$\psi(\mathbf{r},t)=\sqrt{\alpha/2v}\,\exp\left\{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}-\frac{i}{\hbar}Et\right\},$$

$$E=D_\alpha|\mathbf{p}|^\alpha,\quad 1<\alpha\leqslant 2, \tag{48}$$

where v is the particle velocity, $v=\alpha D_\alpha p^{\alpha-1}$. Indeed, by substituting Eq. (48) into Eq. (45) we find

$$\mathbf{j}=\frac{\mathbf{v}}{v},\quad \mathbf{v}=\alpha D_\alpha|\mathbf{p}^2|^{\alpha/2-1}\mathbf{p}, \tag{49}$$

that is, the vector **j** is the unit vector.

Equations (44), (45), and (47) are the fractional generalization of the well-known equations for the probability current density vector of standard quantum mechanics [21].

VI. CONCLUSIONS

The fractional generalization of the Schrödinger equation has been studied. We have established the formula for integration by parts of the quantum Riesz fractional derivative and used it to prove the Hermiticity of the fractional Hamilton operator. A parity conservation law for fractional quantum mechanics was observed. The time-independent fractional Schrödinger equation was introduced. As physical applications of the time-independent fractional Schrödinger equation we found the energy spectrum and equation for the orbit radius of a hydrogenlike atom, the fractional Bohr atom. The energy spectrum of the 1D fractional oscillator was obtained in the semiclassical approximation.

The generalization of the fractional probability current density was derived and discussed.

The generalized equations (8), (22), (29)–(33), (39), (44), (45), and (47) are transformed into the well-known equations of conventional quantum mechanics if we put the Lévy index $\alpha=2$. In other words, fractional quantum mechanics includes standard quantum mechanics as the particular Gaussian case at $\alpha=2$. The quantum mechanical path integral over the Lévy paths at $\alpha=2$ becomes the well-known Feynman path integral and the fractional Schrödinger equation becomes the Schrödinger equation.

The fractional Schrödinger equation provides us with a general point of view on the relationship between the statistical properties of the quantum mechanical path and the structure of the fundamental equations of quantum mechanics.

[1] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[2] R. P. Feynman, *Statistical Mechanics* (Benjamin. Reading, MA, 1972).

[3] P. Lévy, *Théorie de l'Addition des Variables Aléatoires* (Gauthier-Villars, Paris, 1937).

[4] A. Y. Khintchine and P. Lévy, C. R. Hebd. Seances Acad. Sci. **202**, 374 (1936).

[5] W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1966).

[6] V. M. Zolotarev, *One-Dimensional Stable Distributions*, Vol. 65, *Translations of Mathematical Monographs* (AMS, Providence, RI, 1986).

[7] M. Kac, in *Second Berkeley Symposium on Mathematical Statistics and Probability*, edited by Jerzy Neyman (University of California Press, Berkeley, CA, 1951).

[8] E. W. Montroll, in *Physical Reality and Mathematical Description*, edited by C. Enz and M. Mehra (Reidel, Dordrecht, The Netherlands, 1974), pp. 501–598.

[9] B. West, J. Phys. Chem. B **104**, 3830 (2000).

[10] N. Laskin, Phys. Lett. A **268**, 298 (2000).

[11] N. Laskin, Phys. Rev. E **62**, 3135 (2000).

[12] A. I. Saichev and G. M. Zaslavsky, Chaos **7**, 753 (1997).

[13] K. B. Oldham and J. Spanier, *The Fractional Calculus* (Academic, New York, 1974).

[14] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley, New York, 1993).

[15] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications* (Gordon and Breach, Amsterdam, 1993).

[16] I. Podlubny, *Fractional Differential Equations* (Academic, New York, 1999).

[17] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).

[18] R. N. Mantegna and H. E. Stanley, Phys. Rev. Lett. **73**, 2946 (1994).

[19] H. Kleinert, e-print cond-mat/0202311.

[20] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics, and Financial Markets*, 3rd ed. (World Scientific, Singapore, 2002).

[21] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics (Non-relativistic Theory) Course of Theoretical Physics Vol. 3* (Pergamon, London, 1958).

[22] M. Riesz, Acta Math. **81**, 1 (1949).

[23] L. D. Landau and E. M. Lifshitz, *Mechanics, Course of Theoretical Physics Vol. 1, 3rd ed.* (Pergamon, Oxford, 1976).

[24] N. Bohr, Philos. Mag. **26**, 1 (1913); **26**, 476 (1913); **26**, 857 (1913).

[25] N. Bohr, *Collected Works*, edited by J. Rud Nielsen (North-Holland, Amsterdam, 1977), Vol. 4.

[26] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill, New York, 1955), Vol. 3.